Fourier Transform of a Linearly-Chirped Gaussian Pulse

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A linearly-chirped electric field has a frequency that changes linearly with time

$$\omega(t) = \omega_0 + \frac{d\omega}{dt}t = \omega_0 + bt$$

A Gaussian pulse with a frequency chirp can therefore be represented by

$$E(t) = E_0 e^{-at^2} e^{-i\omega(t)t} = E_0 e^{-at^2 - i(\omega_0 + bt)t} = E_0 e^{-(a+ib)t^2 - i\omega_0 t}$$

with a corresponding intensity of

$$I(t) = E(t)E^*(t) = \left(E_0 e^{-(a+ib)t^2 - i\omega_0 t}\right)\left(E_0 e^{-(a-ib)t^2 + i\omega_0 t}\right) = E_0^2 e^{-2at^2}$$

To find the spectrum of this pulse we take the Fourier transform, defined as

$$\tilde{f}(\omega) = \mathcal{F}[f(t)] \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t}\, dt$$

Plugging in the electric field above,

$$\begin{aligned}
\tilde{E}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} E_0 e^{-(a+ib)t^2 - i\omega_0 t} e^{i\omega t}\, dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} E_0 e^{-[(a+ib)t^2 - i(\omega - \omega_0)t]}\, dt
\end{aligned}$$

To solve this integral, we can complete the square in the argument of the exponent:

$$(\alpha t + \beta)^2 = \alpha^2 t^2 + 2\alpha\beta t + \beta^2 = (a+ib)t^2 - i(\omega - \omega_0)t + \beta^2$$

Since the is to be true for all t, we can equate the coefficients

$$\begin{gathered}
\alpha^2 = a + ib \rightarrow \alpha = \sqrt{a+ib} \\
2\alpha\beta = -i(\omega - \omega_0) \rightarrow \beta = -\frac{i(\omega-\omega_0)}{2\alpha} = -\frac{i(\omega - \omega_0)}{2\sqrt{a+ib}}
\end{gathered}$$

and then rewrite the integrand

$$\begin{aligned}
\tilde{E}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} E_0 e^{-[(a+ib)t^2 - i(\omega-\omega_0)t + \beta^2 - \beta^2]}\, dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} E_0 e^{-\left[(a+ib)t^2 - i(\omega-\omega_0)t + (-\frac{i(\omega-\omega_0)}{2\sqrt{a+ib}})^2 - (-\frac{i(\omega-\omega_0)}{2\sqrt{a+ib}})^2\right]}\, dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} E_0 e^{-\left(\sqrt{a+ib}\,t - \frac{i(\omega-\omega_0)}{2\sqrt{a+ib}}\right)^2} e^{\frac{-(\omega-\omega_0)^2}{4(a+ib)}}\, dt \\
&= \frac{1}{\sqrt{2\pi}} E_0 e^{\frac{-(\omega-\omega_0)^2}{4(a+ib)}} \int_{-\infty}^{\infty} e^{-\left(\sqrt{a+ib}\,t - \frac{i(\omega-\omega_0)}{2\sqrt{a+ib}}\right)^2}\, dt
\end{aligned}$$

Now, we change variables

$$
\begin{aligned}
u &\equiv \sqrt{a+ib}\,t - \frac{i(\omega-\omega_0)}{2\sqrt{a+ib}} \\
\frac{du}{dt} &= \sqrt{a+ib}
\end{aligned}
$$

So we have

$$\tilde{E}(\omega) = \frac{1}{\sqrt{2\pi}} E_0 e^{\frac{-(\omega-\omega_0)^2}{4(a+ib)}} \int_{-\infty-\Im[\frac{i(\omega-\omega_0)}{2\sqrt{a+ib}}]}^{\infty-\Im[\frac{i(\omega-\omega_0)}{2\sqrt{a+ib}}]} e^{-u^2} \frac{du}{\sqrt{a+ib}}$$

which is a complex integral over the line $\Im[u] = -\Re\left[\frac{(\omega-\omega_0)}{2\sqrt{a+ib}}\right]$. Because the integrand, e^{-u^2}, is everywhere analytic, we can use Cauchy's Theorem to move the integral to the real axis. Cauchy's Theorem states that the integral of a complex function that is analytic in the domain D over the piecewise-smooth closed contour C in D is 0:

$$\oint_c f(z)\,dz = 0$$

so in our case, tracing out a box,

$$\oint_c e^{-u^2}\,du = \int_{-p}^{p} e^{-u^2}\,du + \int_{p}^{p+iq} e^{-u^2}\,du + \int_{p+iq}^{-p+iq} e^{-u^2}\,du + \int_{-p+iq}^{-p} e^{-u^2}\,du = 0$$

Looking at the integrals over the line $\Re[u] = p$, we can make the substitution

$$
\begin{aligned}
u &= p + ix \\
\frac{du}{dx} &= i
\end{aligned}
$$

and so the integral becomes

$$\int_{p}^{p+iq} e^{-u^2}\,du = \int_0^q i e^{-(p+ix)^2}\,dx = i\int_0^q e^{-p^2} e^{-2ipx} e^{x^2}\,dx$$

taking the limit as p goes to infinity of the integrand,

$$\lim_{p\to\infty} e^{-p^2} e^{-2ipx} e^{x^2} = 0$$

because the second and third terms are finite, and the first term goes to zero. Because the integrand goes to zero, we know

$$\lim_{p\to\infty} \int_p^{p+iq} e^{-u^2}\,du = 0$$

with a similar result for the line $\Re[u] = -p$. Now our contour integral becomes

$$
\begin{aligned}
0 &= \lim_{p\to\infty}\left(\int_{-p}^{p} e^{-u^2}\,du + \int_{p}^{p+iq} e^{-u^2}\,du + \int_{p+iq}^{-p+iq} e^{-u^2}\,du + \int_{-p+iq}^{-p} e^{-u^2}\,du\right) \\
&= \int_{-\infty}^{\infty} e^{-u^2}\,du + \int_{\infty+iq}^{-\infty+iq} e^{-u^2}\,du \\
\int_{-\infty}^{\infty} e^{-u^2}\,du &= \int_{-\infty+iq}^{\infty+iq} e^{-u^2}\,du
\end{aligned}
$$

and so we can rewrite our electric field spectrum as an integral on the real axis

$$\tilde{E}(\omega) = \frac{1}{\sqrt{2\pi}\sqrt{a+ib}} E_0 e^{\frac{-(\omega-\omega_0)^2}{4(a+ib)}} \int_{-\infty}^{\infty} e^{-u^2}\,du$$

What remains now is to evaluate the integral. We start with

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-x^2}\,dx &= \sqrt{\int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-x^2}\,dx} \\
&= \sqrt{\int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy} \\
&= \sqrt{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy}
\end{aligned}$$

and change to polar coordinates (r, θ)

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-x^2}\,dx &= \sqrt{\int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2} r\,dr\,d\theta} \\
&= \sqrt{2\pi \int_{0}^{\infty} e^{-r^2} r\,dr} \\
&= \sqrt{2\pi \left.\frac{e^{-r^2}}{-2}\right|_{0}^{\infty}} = \sqrt{2\pi\left(\frac{1}{2}\right)} \\
&= \sqrt{\pi}
\end{aligned}$$

And so finally the electric field spectrum is

$$\tilde{E}(\omega) = \frac{1}{\sqrt{2}\sqrt{a+ib}} E_0 e^{\frac{-(\omega-\omega_0)^2}{4(a+ib)}}$$

and the spectral intensity is

$$\begin{aligned}
I(\omega) = \tilde{E}(\omega)\tilde{E}^*(\omega) &= \left(\frac{1}{\sqrt{2}\sqrt{a+ib}} E_0 e^{\frac{-(\omega-\omega_0)^2}{4(a+ib)}}\right)\left(\frac{1}{\sqrt{2}\sqrt{a-ib}} E_0 e^{\frac{-(\omega-\omega_0)^2}{4(a-ib)}}\right) \\
&= \frac{1}{2\sqrt{a^2+b^2}} E_0^2 e^{\frac{-(\omega-\omega_0)^2}{4(a+ib)} + \frac{-(\omega-\omega_0)^2}{4(a-ib)}} \\
&= \frac{1}{2\sqrt{a^2+b^2}} E_0^2 e^{\frac{-a(\omega-\omega_0)^2}{2(a^2+b^2)}}
\end{aligned}$$

We can write the FWHM pulse length and spectrum for these Gaussian intensities:

$$e^{-\rho\,\frac{\Delta x}{2}^2} = \frac{1}{2} \rightarrow \Delta x = 2\sqrt{\frac{\ln 2}{\rho}}$$

and for our gaussians

$$\rho_{time} = 2a \qquad\qquad \rho_{freq} = \frac{a}{2(a^2+b^2)}$$

and therefore

$$\Delta t_{FWHM} = \sqrt{\frac{2\ln 2}{a}} \qquad \Delta\omega_{FWHM} = 2\sqrt{2\ln 2}\sqrt{a\left[1+\left(\frac{b}{a}\right)^2\right]} \qquad \Delta\nu_{FWHM} = \frac{\sqrt{2\ln 2}}{\pi}\sqrt{a\left[1+\left(\frac{b}{a}\right)^2\right]}$$