

Photons in a Gaussian Pulse

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Beginning with a 3-dimensional gaussian EM field in vacuum, propagating along the z axis, with $1/e$ radii ($1/e^2$ radii in intensity) in the x and y dimensions of w_x and w_y , respectively, and a $1/e$ half-width temporally of Δt ,

$$\mathbf{E}(x, y, z, t) = E_0 e^{-\frac{x^2}{w_x^2}} e^{-\frac{y^2}{w_y^2}} e^{-\frac{(kz-\omega t)^2}{(\omega \Delta t)^2}} e^{i(kz-\omega t)} \hat{x} \quad (1)$$

$$\mathbf{B}(x, y, z, t) = E_0 e^{-\frac{x^2}{w_x^2}} e^{-\frac{y^2}{w_y^2}} e^{-\frac{(kz-\omega t)^2}{(\omega \Delta t)^2}} e^{i(kz-\omega t)} \hat{y} = \mathbf{H}(x, y, z, t) \quad (2)$$

we can use the time-averaged Poynting vector to determine the power passing through a certain point at a certain time

$$\mathbf{S}(x, y, z, t) = \frac{1}{2} \frac{c}{4\pi} \mathbf{E} \times \mathbf{H}^* = \frac{cE_0^2}{8\pi} e^{-\frac{2x^2}{w_x^2}} e^{-\frac{2y^2}{w_y^2}} e^{-\frac{(kz-\omega t)^2}{(\omega \Delta t)^2}} \hat{z} \quad (3)$$

and can find the total energy in the pulse by integrating the power flow across the $z = 0$ plane at all times t

$$\begin{aligned} E_{tot} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{S}(x, y, 0, t) \cdot \hat{z} dx dy dt \\ &= \frac{cE_0^2}{8\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{2x^2}{w_x^2}} e^{-\frac{2y^2}{w_y^2}} e^{-\frac{2t^2}{\Delta t^2}} dx dy dt \\ &= \frac{cE_0^2}{8\pi} \sqrt{\frac{\pi}{2}} w_x \sqrt{\frac{\pi}{2}} w_y \sqrt{\frac{\pi}{2}} \Delta t \end{aligned} \quad (4)$$

Making use of

$$a_0 \equiv \frac{eE_0}{mc\omega} \quad r_e \equiv \frac{e^2}{mc^2} \quad \lambda_c \equiv \frac{h}{mc} \quad \omega = \frac{2\pi c}{\lambda}$$

we can say

$$\begin{aligned} E_{tot} &= \frac{\frac{cE_0^2}{8\pi} \left(\sqrt{\frac{\pi}{2}}\right)^3 w_x w_y \Delta t \left(a_0^2 \frac{m^2 c^2 \omega^2}{e^2 E_0^2}\right) \left(\hbar \frac{2\pi}{h}\right)}{\left(r_e \frac{mc^2}{e^2}\right) \left(\lambda_c \frac{mc}{h}\right) \left(\omega \frac{\lambda}{2\pi c}\right)} \\ &= \frac{\left(\sqrt{\frac{\pi}{2}}\right)^5 w_x w_y c \Delta t a_0^2 \omega \hbar}{r_e \lambda_c \lambda} \end{aligned} \quad (5)$$

and so, to get the total number of photons, we divide by the photon energy

$$\boxed{N_\gamma = \frac{E_{tot}}{\hbar\omega} = \frac{a_0^2 \left(\sqrt{\frac{\pi}{2}}\right)^5 w_x w_y c \Delta t}{r_e \lambda_c \lambda}} \quad (6)$$