Photons in a Gaussian Pulse

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Beginning with a 3-dimensional gaussian EM field in vacuum, propagating along the z axis, with $^1/_e$ radii ($^1/_{e^2}$ radii in intensity) in the x and y dimensions of w_x and w_y, respectively, and a $^1/_e$ half-width temporally of Δt,

$$\mathbf{E}(x,y,z,t) = E_0 e^{-\frac{x^2}{w_x^2}} e^{-\frac{y^2}{w_y^2}} e^{-\frac{(kz-\omega t)^2}{(\omega\Delta t)^2}} e^{i(kz-\omega t)} \hat{x} \tag{1}$$

$$\mathbf{B}(x,y,z,t) = \frac{E_0}{c} e^{-\frac{x^2}{w_x^2}} e^{-\frac{y^2}{w_y^2}} e^{-\frac{(kz-\omega t)^2}{(\omega\Delta t)^2}} e^{i(kz-\omega t)} \hat{y} = \mu_0 \mathbf{H}(x,y,z,t) \tag{2}$$

we can use the time-averaged Poynting vector to determine the power passing through a certain point at a certain time

$$\mathbf{S}(x,y,z,t) = \tfrac{1}{2}\mathbf{E}\times\mathbf{H}^* = \frac{E_0^2}{2\mu_0 c} e^{-\frac{2x^2}{w_x^2}} e^{-\frac{2y^2}{w_y^2}} e^{-\frac{(kz-\omega t)^2}{(\omega\Delta t)^2}} \hat{z} \tag{3}$$

and can find the total energy in the pulse by integrating the power flow across the $z=0$ plane at all times t

$$\begin{aligned} E_{tot} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{S}(x,y,0,t)\cdot\hat{z}\,dx\,dy\,dt \\ &= \frac{E_0^2}{2\mu_0 c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{2x^2}{w_x^2}} e^{-\frac{2y^2}{w_y^2}} e^{-\frac{2t^2}{\Delta t^2}}\,dx\,dy\,dt \\ &= \frac{E_0^2}{2\mu_0 c}\sqrt{\frac{\pi}{2}}w_x\sqrt{\frac{\pi}{2}}w_y\sqrt{\frac{\pi}{2}}\Delta t \end{aligned} \tag{4}$$

Making use of

$$a_0 \equiv \frac{eE_0}{mc\omega} \qquad r_e \equiv \frac{e^2}{4\pi\epsilon_0 mc^2} \qquad \lambda_c \equiv \frac{h}{mc} \qquad \mu_0\epsilon_0 \equiv \frac{1}{c^2} \qquad \omega = \frac{2\pi c}{\lambda}$$

we can say

$$\begin{aligned} E_{tot} &= \frac{\frac{E_0^2}{2\mu_0 c}\left(\sqrt{\frac{\pi}{2}}\right)^3 w_x w_y \Delta t \left(a_0^2\frac{m^2c^2\omega^2}{e^2E_0^2}\right)\left(\hbar\frac{2\pi}{h}\right)\left(\epsilon_0\mu_0 c^2\right)}{\left(r_e\frac{4\pi\epsilon_0 mc^2}{e^2}\right)\left(\lambda_c\frac{mc}{h}\right)\left(\omega\frac{\lambda}{2\pi c}\right)} \\ &= \frac{\left(\sqrt{\frac{\pi}{2}}\right)^5 w_x w_y c\Delta t\, a_0^2\omega\hbar}{r_e\lambda_c\lambda} \end{aligned} \tag{5}$$

and so, to get the total number of photons, we divide by the photon energy

$$\boxed{N_\gamma = \frac{E_{tot}}{\hbar\omega} = \frac{a_0^2\left(\sqrt{\frac{\pi}{2}}\right)^5 w_x w_y c\Delta t}{r_e\lambda_c\lambda}} \tag{6}$$